LIMITING EIGENVALUE DISTRIBUTION FOR BAND RANDOM MATRICES

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An equation is obtained for the Stieltjes transform of the normalized eigenvalue distribution of band random matrices in the limit in which the band width and rank of the matrix simultaneously tend to infinity. Conditions under which this limit agrees with the semicircle law are found.

1. STATEMENT OF PROBLEM, FORMULATION OF RESULTS, AND DISCUSSION

Random symmetric, or Hermitian matrices arise in many branches of physics and mathematics (see, for example, the reviews [1—5] and the references there). Among the many and numerous problems associated with the properties of such matrices, one of the simplest and, at the same time, most important is that of the distribution of their eigenvalues. As was first shown by Wigner [6], in the case of matrices with independent Gaussian elements this problem can be solved exactly in the limit of infinite rank of these matrices. Namely, let the symmetric matrix $W^{(n)}$ of rank $n = 2m + 1$ have elements of the form

$$W^{(n)}(x, y)=n^{-1/2}V(x, y), \quad |x|, |y|\leq m, \tag{1.1}$$

where $V(x, y) = V(y, x)$ are independent (for $x \leq y$) Gaussian variables such that

$$E\{V(x, y)\}=0, \quad E\{V^2(x, y)\}=V^2. \tag{1.2}$$

We denote by $\lambda_{-m}, \ldots, \lambda_m$ the eigenvalues of $W^{(n)}$, and let

$$N_n(\lambda)=n^{-1}\sum_{\lambda_k<\lambda} 1 \tag{1.3}$$

be the function, normalized by the rank of the matrix, that counts its eigenvalues. Then in accordance with [6]

$$\lim E\{N_n(\lambda)\}=N_W(\lambda), \tag{1.4}$$

where the nondecreasing function $N_W(\lambda)$ is differentiable and its derivative — the density of states — has the form

$$n_W(\lambda)=N_W'(\lambda)=\begin{cases}\dfrac{1}{2\pi V^2}\sqrt{4V^2-\lambda^2}, & |\lambda|\leq 2V,\\ 0, & |\lambda|>2V.\end{cases} \tag{1.5}$$

This limit function is called the Wigner distribution, or the semicircle law.

Wigner's proof [6] was based on the moment method. Namely, he showed that for any $k = 0, 1, \ldots$ there exist the limits

$$\lim_{n\to\infty} n^{-1}E\{\mathrm{Tr}(W^{(n)})^k\}=\lim_{n\to\infty} n^{-1}E\Big\{\sum_{|l|\leq m}\lambda_l^k\Big\}\equiv m_k$$

of the moments of the function (1.3). Here, $m_k = V^{2k}M_k$, where M_k is the number of different root trees (graphs without cycles) containing k links. The generating function of the numbers M_k is readily found and gives the semicircle law.

This method can be used not only for Gaussian matrices, but it requires the existence of all moments of the quantities $V(x, y)$ (as in the analogous situation in the central limit theorem). In particular, it can be readily applied to band matrices, which are defined below in (1.7)—(1.9).

Experience gained in probability theory teaches us that it is much more convenient to work with generating functions rather than the moments of random variables (in fact, generating functions also make their appearance in Wigner's work). It was noted in [7] that in the given context the generating function

$$n^{-1}\sum_{k=0}^{\infty} z^{-k-1}\,\mathrm{Tr}(W^{(n)})^k=n^{-1}\,\mathrm{Tr}(W^{(n)}-zI)^{-1}, \quad \mathrm{Im}\, z\neq 0,$$

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which is generated by the resolvent $(W^{(n)} - zI)^{-1}$ of the random matrix, is particularly conveneint. The use of this generating function made it possible to show that the semicircle law arises as the limit for the eigenvalue distribution of symmetric matrices with any elements (not necessarily equally distributed) that are independent for $x \le y$ and in addition to (1.2) satisfy the following condition: for any $c > 0$

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{|x|,\,|y|\le m} \int_{|V|\ge c\sqrt{n}} V^2\, dP\{V(x,y) \le V\} = 0. \tag{1.6}$$

Under this condition, $N_n(\lambda)$ in (1.3) converges to $N_W(\lambda)$ not only in the mean (as in (1.4)), but also in probability [4,5].

The condition (1.6) is the natural analog of Lindeberg's condition in probability theory, the necessary and sufficient condition for the validity of the central limit theorem. In accordance with [5], the condition (1.6) is also necessary for convergence of the function (1.3) to (1.4)—(1.5) in probability. This enables us to conclude that the semicircle law (1.5) is as universal a limit form of the eigenvalue distribution of random symmetric matrices with independent (except for the symmetry condition) elements as the normal law is the universal limit of the distribution of sums of independent random variables.

In this paper, we shall consider an ensemble of random matrices more general than (1.1), (1.2), namely, those for which the elements have the form

$$W_1^{(n)}(x,y) = b_n^{-1/2}\, V(x,y)\, v\left(\frac{x-y}{b_n}\right). \tag{1.7}$$

Here, $0 < b_n \le n$ is the sequence of integers such that $b_n \to \infty$ and there exists the finite or infinite limit

$$2\nu = \lim_{n\to\infty} \frac{n}{b_n} \ge 1, \tag{1.8}$$

with $V(x, y) = V(y, x)$ a family of equally distributed and independent, for $x \le y$, random variables satisfying (1.2).

The function $v(t)$, $|t| \le 2\nu$, is a piecewise continuous even function of compact support for which

$$\sup_{|t|\le 2\nu} |v(t)| \le V_0, \qquad \int_{-2\nu}^{2\nu} v^2(t)\,dt = 1. \tag{1.9}$$

In particular, if

$$v(t) = \chi_1(t), \tag{1.10}$$

where $\chi_1(t)$ is the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$, then the matrix elements (1.7) are nonvanishing only in a "band" of widths b_n symmetric with respect to the principal diagonal.

If b_n does not depend on n (and is finite), then we have a finite-difference operator of order $\beta + 1 = (b + 1)/2$. Here, even in the case $b = 1$, i.e., in the case of operators of second order (Jacobi matrices), the calculation of the limit $N_n(\lambda)$, which is called the integrated density of states, requires the solution of a certain integral equation, and this can be done only in a few cases (see, for example, [8] and the references given there). But if $b_n \to \infty$ as $n \to \infty$ and the limit (1.8) is equal to infinity, then the number $b_n n$ of nonvanishing elements is infinitesimally small compared with the number n^2 of nonvanishing elements of the Wigner matrices (1.1)—(1.2), although it does grow more rapidly than n, i.e., the number of nonvanishing elements of the Jacobi matrices.

We now formulate the main result of the paper.

THEOREM 1. *The normalized counting function $N_n(\lambda)$ in (1.3) for the ensemble (1.7)—(1.9) for every λ converges in probability to a nonrandom limit integrated density of states $N(\lambda)$ that is an absolutely continuous function whose derivative (the density of states) is bounded. At the same time*

a) if the number ν in (1.8) is equal to infinity, then $N(\lambda)$ is identical with the semicircle law (1.5);

b) if the number ν in (1.8) is finite, then $N(\lambda)$ agrees with the semicircle law if and only if the function $v^2(t)$ in (1.7) is the restriction to the interval $(-2\nu, 2\nu)$ of an even 2ν-periodic function.

Thus, if the width of the band of nonvanishing matrix elements (1.7) grows more slowly than the rank of the matrices the limit integrated density of states is always the Wigner density. But if the number of nonvanishing elements of these matrices has the same order as for Wigner matrices, then the semicircle law arises if and only if these matrices are cyclic, or, in terms of statistical physics, in the case when the boundary conditions are cyclic and not zero-value conditions.

It is here appropriate to mention that in the recent [9] a numerical analysis of band matrices of the form (1.7), (1.10) and certain heuristic arguments provided the basis for a very interesting conjecture, according to which the spectral properties of such matrices and, in the first place, the property of localization of their states are determined by the parameter $x = b_n^2/n$ (the numerical results demonstrate a clearly expressed localization when $x << 1$ and almost complete absence when $x >> 1$). In the interpretation of [9], this parameter is analogous to the localization length (reciprocal Lyapunov exponent) in the theory of localization (for a discussion of this last, see, for example, [8]).

In accordance with our (rigorous) results, this parameter does not play a special role in the formation of the density of states of the considered matrices. However, we are not inclined to regard our results as incompatible with the interesting conjecture made in [9], since, as is well known, the integrated density of states has very little sensitivity to the localization properties of the states of random operators (especially when $b > 1$).

In what follows, we shall need the Stieltjes transform $f(z)$, $\operatorname{Im} z \neq 0$, of a nondecreasing function $\mu(\lambda)$, $\mu(-\infty) = 0$, $\mu(+\infty) = 1$; it is defined as

$$f(z)=\int \frac{d\mu(\lambda)}{\lambda-z}, \quad \operatorname{Im} z\neq 0. \tag{1.11}$$

The function $f(z)$ is obviously analytic for nonreal z and satisfies the inequalities

$$|f(z)|\leq|\operatorname{Im} z|^{-1}, \quad \operatorname{Im} f(z)\operatorname{Im} z>0, \quad \operatorname{Im} z\neq 0. \tag{1.12}$$

The second of these inequalities defines the class of functions called Nevanlinna functions. The function $\mu(\lambda)$ can be recovered from $f(z)$ by means of the Stieltjes—Perron inversion formula

$$\mu(\lambda_1)-\mu(\lambda_2)=\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{\lambda_2}^{\lambda_1}\operatorname{Im} f(\lambda+i\varepsilon)\,d\lambda, \tag{1.13}$$

where λ_1 and λ_2 are points of continuity of $\mu(\lambda)$.

We now formulate the main technical result, from which Theorem 1 can be readily reduced.

THEOREM 2. *Let $r_n(z)$ be the Stieltjes transform of the function (1.3). Then there exists a nonrandom Nevanlinna function $r(z)$ such that for all nonreal z*

$$p\text{-}\lim_{n\to\infty} |r_n(z)-r(z)|=0. \tag{1.14}$$

At the same time:

a) if in the relation (1.8) $\nu = \infty$, then

$$V^2r^2+rz+1=0, \tag{1.15}$$

where the parameter V is determined by (1.2);

b) if in the relation (1.8) $\nu < \infty$, then

$$r(z)=\frac{1}{2\nu}\int_{-\nu}^{\nu} r(t;z)\,dt, \tag{1.16}$$

where the function $r(t; z)$, $|t| \leq \nu$, $\operatorname{Im} z \neq 0$, is the unique solution of the equation

$$r(t;z)=-\left(z+V^2\int_{-\nu}^{\nu} v^2(t-t')r(t';z)\,dt'\right)^{-1}. \tag{1.17}$$

*that for every t satisfies the inequalities (1.2).**

We deduce Theorem 1 from Theorem 2. The existence of the limit integrated density of states $N(\lambda)$ follows from (1.14) and a lemma in [7] that is, roughly speaking, a generalization of Helly's well-known theorems to the case of convergence in probability.

To establish that in case a) of Theorem 1 we obtain the semicircle law, we note that the solution of Eq. (1.15) in the class (1.12) has the form

$$r_W(z)=\frac{-z+\sqrt{z^2-4V^2}}{2V^2}, \tag{1.18}$$

in which the branch of the function having asymptotic behavior z as $|z| \to \infty$ is chosen. From this and from (1.13) we obtain (1.5).

*A similar result is in the book [5].

We now turn to the case of finite ν. Here, if the function $v^2(t)$ is 2ν-periodic, then Eq. (1.7) also has a t-independent solution $r(z)$, which is obviously identical to (1.18). This again leads to (1.5).

Now suppose $v^2(t)$ does not possess this property. Then, as is readily seen, the function

$$u(t)\equiv\int_{-\nu}^{\nu} v^2(t-t')\,dt'$$

cannot be a constant on the interval $(-\nu, \nu)$. It follows from this that the first three terms of the expansion of $r(z)$ in inverse powers of z have the form

$$-\frac{1}{z}-\frac{a_1}{z^3}-\frac{a_2}{z^5},\qquad a_1=\frac{1}{2\nu}\int_{-\nu}^{\nu}u(t)\,dt,\qquad a_2=\frac{1}{\nu}\int_{-\nu}^{\nu}u^2(t)\,dt.$$

Hence and from the Cauchy—Schwarz inequality we obtain the rigorous inequality

$$a_1^2a_2^{-1}<2.$$

On the other hand, for the semicircle law [for which $u(t)\equiv$ const, $|t| \leq \nu$], $a_1^2a_2^{-1} = 2$. Therefore, in the considered case of nonperiodic $v^2(t)$ the integrated density of states cannot be the semicircle law.

In [4] (see also [5]) there was considered an ensemble of random matrices more general than (1.1)—(1.2), namely, of the form

$$h_0^{(n)}+W^{(n)}, \tag{1.19}$$

where the "unperturbed" matrix $h_0^{(n)}$ is diagonal and its eigenvalues $\mu_{-m}^{(n)}, \ldots, \mu_m^{(n)}$ are such that the counting function analogous to (1.3), namely,

$$N_0^{(n)}(\lambda)=n^{-1}\sum_{\mu_l\leqslant\lambda}1 \tag{1.20}$$

converges in the limit $n\to\infty$ to some limit function $N_0(\lambda)$ at every continuity point of this function. Then if

$$r_0(z)=\int\frac{dN_0(\lambda)}{\lambda-z} \tag{1.21}$$

is the Stieltjes transform of this integrated density of states of the unperturbed operator $h_0^{(n)}$ the Stieltjes transform $r(z)$ of the total operator (1.19) is the unique solution in the class (1.12) of the functional equation

$$r(z)=r_0(z+V^2r(z)). \tag{1.22}$$

In the case $r_0 = -z^{-1}$, corresponding to $h_0^{(n)} \equiv 0$, this equation obviously reduces to (1.15).

We now briefly describe the results that can be obtained for an ensemble of the form

$$h_0^{(n)}+W_1^{(n)}, \tag{1.23}$$

which bears the same relation to (1.7) as the ensemble (1.19) does to (1.1) (the corresponding proofs will be given in a separate publication). These results can be obtained by a generalization of the method used below to analyze the ensemble (1.7).

We consider first the case when the function $v(t)$ has the form (1.10) and $b = n$, i.e., the case when the perturbation is a matrix of the form (1.1), (1.2). In this case, the result is given by the same formula (1.22), which in [4] was proved for diagonal $h_0^{(n)}$. In the considered general case, in which $h_0^{(n)}$ is, in general, nondiagonal, the requirement on $h_0^{(n)}$ remains the same as in [4], i.e., one requires only the existence of the limit integrated density of states. This requirement is satisfied, for example, when $h_0^{(n)}$ is the restriction to the interval $|x|$, $|y| \leq m$ of the matrix of a metrically transitive operator that does not depend on $W^{(n)}$ (concerning such operators, see, for example, the review [10]). An important special case corresponds to Toeplitz $h_0^{(n)}$. Thus, the deformed semicircle law is valid not only for diagonal $h_0^{(n)}$ but also in the much more general case of nondiagonal $h_0^{(n)}$ possessing a limit integrated density of states.

Now suppose $v(t)$ is an arbitrary function that satisfies the conditions (1.9) and $b = o(n)$, i.e., in (1.8) $\nu = \infty$. In this case, formula (1.22) can be established for diagonal $h_0^{(n)}$ whose elements form an ergodic sequence that does not depend on $W_1^{(n)}$ and, in the case of Toeplitz $h_0^{(n)}$, whose elements depend on the difference of the indices. However, formula (1.22) cannot be true for all $h_0^{(n)}$ even if a restriction is made to diagonal matrices. Thus, if

$$\mu_i^{(n)}=\mu\left(\frac{i}{n}\right),$$

where $\mu(t)$, $|t| \le 1/2$, is a piecewise continuous function, then the result has the form

$$r(z)=\int_{-1/2}^{1/2} r_W(z-\mu(t))dt=\int r_W(z-\mu)N_0(d\mu), \tag{1.24}$$

where

$$N_0(\mu)=|\{t\in[-1/2,1/2]:\mu(t)\le\mu\}|$$

is the unperturbed density of states. From this and (1.13) it follows that the corresponding integrated density of states has the form

$$N(\lambda)=\int N_W(\lambda-\mu)dN_0(\mu), \tag{1.25}$$

i.e., is the convolution of the semicircle law and the unperturbed integrated density of states. Such a formula is typical for random operators that, besides the "macroscopic" length scale n, contain a further length scale which is large compared with the "interatomic" distance but small compared with the macroscopic scale (see [8]). In our case, b plays the role of this second scale, since our condition (1.7) for $\nu = \infty$ can be understood as the mathematical expression of the inequalities

$$1\ll b\ll n.$$

Similar but less simply formulated results can also be obtained in some other cases.

2. PROOF OF THEOREM 2 FOR GAUSSIAN MATRICES

To make the proof more transparent, we begin with matrices (1.7) whose elements are Gaussian random variables satisfying the conditions (1.2). We need the two following facts.

1. If ξ is a Gaussian random variable, $E\{\xi\} = 0$, $E\{\xi^2\} = V^2$, and $f(\xi)$ is a boundedly differentiable function, then

$$E\{\xi f(\xi)\}=E\{f'(\xi)\}. \tag{2.1}$$

2. If W is a symmetric real matrix of rank $n = 2m + 1$ with elements $W(x, y)$, $|x|$, $|y| \le m$, and $G(x, y) = (W - z)^{-1}(x, y)$ is the matrix of its resolvent (Green's function), then

$$\frac{\partial G(x,y)}{\partial W(t,s)}=-G(x,t)G(s,y)-G(x,s)G(t,y). \tag{2.2}$$

The relation (2.1) can be proved by integration by parts. The relation (2.2) follows from the resolvent identity

$$(A+B-z)^{-1}=(A-z)^{-1}-(A-z)^{-1}B(A+B-z)^{-1}, \tag{2.3}$$

if in it we set $A = W$, $B = \delta W$, and from the inequalities

$$|(A-z)^{-1}(x,y)|\le\|(A-z)^{-1}\|\le|\operatorname{Im} z|^{-1}, \tag{2.4}$$

which hold for any symmetric matrix A.

We now consider the following sequence of moments of the matrix elements $G_n(x, y)$ of the Green's function $G = (W^{(n)} - z)^{-1}$ of the matrices (1.7):

$$E\{G(x_1,x_1)\ldots G(x_k,x_k)\}\equiv g_k(x_1,\ldots,x_k;\ z). \tag{2.5}$$

Suppose first $k = 1$. Then from the resolvent identity (2.3) for $A = 0$, $B = W^{(n)}$ we obtain

$$g_1(x_1;z)=-z^{-1}+\frac{1}{\sqrt{b}}\sum_{|y|\le m}E\{V(x_1,y)G(y,x_1)\}.$$

Calculating the mathematical expectation in the second term on the right by means of (2.2) and (1.7), we find that

$$g_1(x_1,z)=-z^{-1}-\frac{V^2}{zb}\sum_{|y|\le m}v^2\left(\frac{x_1-y}{b}\right)g_2(y,x_1;z)+\rho_1(x_1;z), \tag{2.6}$$

where

$$\rho_1(x_1;z)=-\frac{V^2}{zb}\sum_{|y|\leqslant m}v^2\left(\frac{x_1-y}{b}\right)E\{G^2(y,x_1)\}. \tag{2.7}$$

By means of the Cauchy—Schwarz inequality (1.9) and (2.4) we can readily show that

$$|\rho_1(x_1;z)|\leqslant\frac{(VV_0)^2}{b\eta^3}, \tag{2.8}$$

where

$$\eta=|\operatorname{Im} z|. \tag{2.9}$$

Similarly, for $k > 1$, using in the first factor in (2.5) the resolvent identity, we obtain

$$g_k(x_1,\ldots,x_k;z)=-z^{-1}g_{k-1}(x_2,\ldots,x_k;z)-\frac{V^2}{zb}\sum_{|y|\leqslant m}v^2\left(\frac{x_1-y}{b}\right)g_{k+1}(y,x_1,\ldots,x_k;z)+\rho_k, \tag{2.10}$$

where

$$|\rho_k(x_1,\ldots,x_k;z)|\leqslant\frac{(VV_0)^2(2k-1)}{b\eta^{k+2}}. \tag{2.11}$$

We shall now regard the relations (2.6) and (2.7) as an infinite system of integral equations for the moments (2.5). For this, we introduce the Banach space $\mathfrak{B}_n$ of sequences $f=\{f_k(x_1,\ldots,x_k)\}_{k=1}^{\infty}$, in which $|x_i| \leq m$, $i = 1, \ldots, k$, and the norm is defined as

$$\|f\|=\sup_{k\geqslant 1}\xi^k\sup_{|\operatorname{Im} z|\geqslant\xi}\max_{|x_1|,\ldots,|x_k|\leqslant m}|f_k(x_1,\ldots,x_k)|, \tag{2.12}$$

where ξ is a fixed number satisfying the inequality

$$2V<\xi<3V. \tag{2.13}$$

By virtue of the inequality (2.4), the sequences g in (2.5) and ρ in (2.8), (2.11) belong to $\mathfrak{B}_n$ if

$$|\operatorname{Im} z|=\eta\geqslant 3V, \tag{2.14}$$

with

$$\|\rho\|\leqslant\frac{c}{b}, \tag{2.15}$$

where c does not depend on n and b.

We define in $\mathfrak{B}_n$ the vector $a=\{-z\delta_{k,1}\}_{k=1}^{\infty}$ and the operator A:

$$(Af_1)(x_1)=-\frac{1}{2b}\sum_{|y|\leqslant m}v^2\left(\frac{x_1-y}{b}\right)f_2(y,x_2), \tag{2.16}$$

$$(Af)_k(x_1,\ldots,x_k)=-\frac{1}{z}f_{k-1}(x_2,\ldots,x_k)-\frac{1}{zb}\sum_{|y|\leqslant m}v^2\left(\frac{x_1-y}{b}\right)f_{k+1}(y,x_1,\ldots,x_k),\qquad k>1. \tag{2.17}$$

Then the relations (2.6) and (2.10) can be written in the form of the following linear equation in $\mathfrak{B}_n$:

$$g=Ag+a+\rho. \tag{2.18}$$

It is easy to show that if $\|f\| = 1$ then

$$|(Af_k)(x_1,\ldots,x_k)|\leqslant\frac{1}{\eta\xi^k}\left(\xi+\frac{1}{\xi}\frac{1}{b}\sum_{|y|\leqslant m}v^2\left(\frac{y}{b}\right)\right)$$

and therefore under the conditions (1.9), (2.13), and (2.14) and for sufficiently large b

$$\|A\|\leqslant\alpha<1. \tag{2.19}$$

Hence and from the relations (2.18) and (2.11) we conclude that if $r=\{r_k(x_1,\ldots,x_k;\ z)\}_{k=1}^{\infty}$ s the unique solution belonging to $\mathfrak{B}_n$ of the equation

$$r^{(n)}=Ar^{(n)}+a, \tag{2.20}$$

then

$$\|g-r^{(n)}\|\leqslant\frac{c}{1-\alpha}\frac{1}{b}. \tag{2.21}$$

In particular, by virtue of (2.12)

$$\lim_{n\to\infty}\max_{|x_1|\leqslant m}|E\{G(x_1,x_1)-r_1^{(n)}(x_1,z)|=0. \tag{2.22}$$

Thus, we have reduced the problem to the system (2.20). This infinite system of linear equations can, in its turn, be reduced by the factorizing ansatz

$$r_k(x_1,\ldots,x_k;z)=\prod_{i=1}^{k}r_1^{(n)}(x_i;z), \tag{2.23}$$

to a single but nonlinear equation for $r_1^{(n)}$:

$$r_1^{(n)}(x)=-\frac{1}{z}-\frac{V^2}{zb}\sum_{|y|\leqslant m}v^2\left(\frac{x-y}{b}\right)r_1^{(n)}(x)r_1^{(n)}(y),$$

or

$$r_1^{(n)}(x)=-\left(z+\frac{V^2}{b}\sum_{|y|\leqslant m}v^2\left(\frac{x-y}{b}\right)r_1^{(n)}(y)\right)^{-1}. \tag{2.24}$$

One can show that the nonlinear operator defined by the right-hand side of the last equation will be contractive in the metric space of sequences $f(x,z)$, $|x|\leqslant m$, $z\in\mathbb{C}\setminus\mathbb{R}$, which for every x are Nevanlinna functions with metric

$$\sup_{|\mathrm{Im}\,z|\geqslant 3V}\max_{|x|\leqslant m}|f_1(x;z)-f_2(x;z)|. \tag{2.25}$$

Therefore, the unique solution of Eq. (2.24) in this space determines by means of (2.23) a solution of the system (2.20) that is unique by virtue of (2.19).

We now divide the interval $(-\nu,\nu)$ into intervals of length $1/b$. Then the sum in (2.24) can be interpreted as an integral sum. This gives the justification for introducing the "limit" (as n, $b\to\infty$) equation

$$r(t;z)=-\left(z+\int_{-\nu}^{\nu}v^2(t-t')r(t';z)dt'\right)^{-1}, \tag{2.26}$$

where the continuous variable ranges over the interval $[-\nu,\nu]$, and the function $r(t;z)$ is piecewise continuous with respect to $t\in[-\nu,\nu]$ for every nonreal z and is a Nevanlinna function with respect to z for each fixed t.

Equation (2.26), like (2.24), can be uniquely solved in the space with metric

$$\sup_{|\mathrm{Im}\,z|\geqslant 3V}\sup_{|t|\leqslant\nu}|f_1(t;z)-f_2(t;z)|. \tag{2.27}$$

It follows from this and from our conditions on the function $v(t)$ in (1.7) that the difference of the solutions of these equations satisfies the limit relation

$$\lim_{n\to\infty}\sup_{|\mathrm{Im}\,z|\geqslant 3V}\sup_{|x|\leqslant m}\sup_{\frac{x}{b}\leqslant t\leqslant\frac{x+1}{b}}|r(t;z)-r_1^{(n)}(x;z)|=0. \tag{2.28}$$

By virtue of this relation and (2.22),

$$\lim_{n\to\infty}\sup_{|\mathrm{Im}\,z|>3V}\left|E\left\{\frac{1}{n}\sum_{|x|\leqslant m}G(x,x)\right\}-\frac{1}{2\nu}\int_{-\nu}^{\nu}r(t;z)dt\right|=0, \tag{2.29}$$

where in the case $\nu=\infty$ the integral is to be understood as

$$\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}\dots dt.$$

Up to now, our results have applied to the cases of both finite and infinite ν in (1.8). Now suppose $\nu = \infty$. Then the solution of Eq. (2.26) does not depend on t and by virtue of (1.15) is identical to (1.18). In addition, by virtue of the spectral theorem

$$r_n=\frac{1}{n}\sum_{|x|\leqslant m} G(x,x) \tag{2.30}$$

is identical to the Stieltjes transform of the function (1.3). Therefore, in the considered case the relation (2.29) is transformed into

$$\lim_{n\to\infty}\sup_{|\mathrm{Im}\, z|\geqslant 3V}|E\{r_n(z)\}-r_W(z)|=0. \tag{2.31}$$

We have proved a weakened form of Theorem 1a, in which convergence in probability is replaced by convergence of mean values.

We now show that there also holds the stronger relation

$$\lim_{n\to\infty}\sup_{|\mathrm{Im}\, z|\geqslant 3V}E\{|r_n(z)-r(z)|^2\}=0, \tag{2.32}$$

from which (1.14) obviously follows.

For this, we consider for all $k,\ l \geq 1$ the sequence of moments

$$g_{k,l}=(x_1,\dots,x_k,y_1,\dots,y_k)=E\{G(x_1,x_1)\dots G(x_k,x_k)\overline{G(y_1,y_1)}\dots\overline{G(y_k,y_k)}\}. \tag{2.33}$$

Arguing essentially as in the derivation of the system (2.6), (2.10), we find that

$$g_{k,l}(x_1,\dots,y_l)=-\frac{1}{z}g_{k-1,l}-\frac{1}{zb}\sum_{|y|\leqslant m}v^2\left(\frac{x_1-y}{b}\right)g_{k+1,l}(y,x_1,\dots,y_l)+\rho_{k,l}, \tag{2.34}$$

where $g_{0,0}=-1/z,\ g_{k,0}=\bar g_k,\ g_{0,l}=\bar g_l$ and (cf. (2.11))

$$|\rho_{k,l}|\leqslant\frac{2V_0^2(k+l)}{3\eta^{k+l+2}}. \tag{2.35}$$

Therefore, regarding again the relations (2.34) as a system of equations in an appropriate Banach space, we find as in the case of (2.21) that its solution differs by an amount of order $O(1/b)$ from the solution of the equation of the same form as (2.34) but without $\rho_{k,l}$ on the right-hand side. Like (2.23), this last equation has the factorizing solution

$$r_{k,l}(x_1,\dots,y_l)=r_1(x_1)\dots r_1(x_k)\overline{r_1(y_1)}\dots r_1(y_l),$$

where, as before, $r_1(x;\ z)$ satisfies (2.24). Therefore, in particular (cf. (2.22))

$$\sup_{|\mathrm{Im}\, z|\geqslant 3V}\ \max_{|x|,|y|\leqslant m}|E\{G(x,x)\overline{G(y,y)}\}-E\{G(x,x)\}E\{\overline{G(y,y)}\}|=O\left(\frac{1}{b}\right). \tag{2.36}$$

It follows from this that for (2.30)

$$\sup_{|\mathrm{Im}\, z|\geqslant 3V}E\{|r_n-E\{r_n\}|^2\}=O\left(\frac{1}{b}\right). \tag{2.37}$$

In order to obtain (2.32), it is now merely necessary to take into account the relation (2.28) which enables us to replace $E\{r_n\}$ in (2.37) by

$$\frac{1}{2\nu}\int_{-\nu}^{\nu}r(t;\ z)\,dt.$$

Theorem 2 is proved for Gaussian matrices.

3. PROOF OF THEOREM 2 UNDER THE CONDITION OF FINITE VARIANCE OF THE MATRIX ELEMENTS

Now suppose the random variables $V(x, y)$ in (2.7) are arbitrary independent (for $x \leq y$) symmetric, $V(x, y) = V(y, x)$, and equally distributed quantities that satisfy the conditions (1.2).

We assume at first that in addition $V(x, y)$ are bounded, i.e.,

$$|V(x, y)| \leqslant C < \infty. \tag{3.1}$$

We show that in this case we again obtain the system of equations (2.6), (2.10) with remainder term ρ_k that has the order $O(b^{-1/2})$. It is easy to show that all the subsequent arguments in Sec. 2 used only this system.

We shall proceed from the following relation, which is obtained from the resolvent identity (2.3) by regarding the fixed element $V(x, y)$ in it as a perturbation:

$$G(t, s) = \hat{G}(t, s) - \frac{1}{\sqrt{b}} (\hat{G} V^{xy} G)(t, s), \tag{3.2}$$

where $\hat{G} = G|_{V(x, y)=0}$ ...... and V^{xy} is a matrix for which only $V(x, y)$ and $V(y, x) = V(x, y)$ are nonvanishing elements. We again consider the system of moments (2.5) and, as in the derivation of (2.10), replace $G(x_1, x_1)$ by means of the identity

$$G(x_1, x_1) = -\frac{1}{z} + \frac{1}{z\sqrt{b}} \sum_{|y| \leqslant m} V(x_1, y) G(y, x_1), \tag{3.3}$$

so that

$$g_k(x_1, \ldots, x_k) = -\frac{1}{z} g_{k-1}(x_2, \ldots, x_k) + \frac{1}{z\sqrt{b}} E\{V(x_1, y) G(y, x_1) G_2 \ldots G_k\}, \tag{3.4}$$

where $G_i = G(x_i, x_i)$. In the second term here on the right, we replace G_2 by its expression from (3.2) for $t = s = x_2$, $x = x_1$. Then this term takes the form

$$\frac{1}{z\sqrt{b}} \sum_y E\{V(x_1, y) G(y, x_1) \hat{G}_2 G_3 \ldots G_k\} - \frac{1}{zb} \sum_y E\{V^2(x_1, y) \times$$

$$G(y, x_1) [\hat{G}(x_2, y) G(x_1, x_2) + \hat{G}(x_2, x_1) G(y, x_2)] G_3 \ldots G_k\}. \tag{3.5}$$

Using the same arguments as in the derivation of (2.8), we can show that the second member in the second term does not exceed

$$\frac{c^2}{b\eta^k} E\left\{\left|\sum_y G(y, x_1) G(x_2, y)\right|\right\} \leqslant \frac{c^2}{b\eta^k} E\{[(GG^*)(x_1, x_2)]^{1/2} [(GG^*)(x_2, x_2)]^{1/2}\} \leqslant \frac{c^2}{b\eta^{k+2}}. \tag{3.6}$$

To estimate the first member, we use (3.2) and express $G(x_2, y)$ in terms of $G(x_2, y)$ and

$$\frac{1}{\sqrt{b}} (\hat{G} V^{xy} G)(x_2, y).$$

The first of the obtained terms can be estimated by means of inequalities analogous to (3.6), and, as is readily seen, the other two are bounded by

$$\frac{1}{b\sqrt{b}} \sum_y \frac{c^3}{\eta^{k+3}} \leqslant \frac{1}{\sqrt{b}} \frac{c^3}{\eta^{k+3}}.$$

Therefore, the replacement in the second term in (3.4) of $G(x_2, x_2)$ by $\hat{G}(x_2, x_2)$, which does not contain $V(x, y)$, leads to an error $2c^2(\sqrt{b}\eta^{k+2})^{-1}$. Similarly, replacement in the obtained expression of G_3 by G_3 gives the same error. Therefore, making such a substitution for all $G_2, G_3, \ldots, G_k$ in (3.4), we find that the second term in (3.3) can be written in the form

$$-\frac{1}{z\sqrt{b}} \sum_y E\{V(x_1, y) G(y, x_1) \hat{G}_2 \ldots \hat{G}_k\} + \rho_k^{(1)}(x_1, \ldots, x_k), \tag{3.7}$$

where

$$|\rho_k^{(1)}| \leqslant \frac{2c^2(k-1)}{b\eta^{k+2}}. \tag{3.8}$$

We now integrate the relation (3.1) once and set $t = y$, $s = x_1$:

$$G(y, x_1)=\hat{G}(y, x_1)-V(x, y)[\hat{G}(y, y)\hat{G}(x_1, x_1)+\hat{G}^2(y, x_1)]+(\hat{G}V^{xy}\hat{G}V^{xy}G)(y, x_1). \tag{3.9}$$

We substitute this expression for $G(y, x_1)$ in (3.7). Then because $E\{V(x, y)\} = 0$ and the remaining factors in the first term in (3.7) do not depend on $V(x, y)$, the first term on the right in (3.9) does not contribute to (3.7), the second makes the contribution

$$\frac{1}{zb}\sum_{|y|\leq m} v^2\left(\frac{x-y}{b}\right)E\{\hat{G}(y, y)\hat{G}_1\ldots\hat{G}_k\},$$

while the remaining two are small in the limit $b\to\infty$. More precisely, arguments like those that led to (2.8), (2.11), and (3.8) show that these terms, which we denote by $\rho_k^{(3)}$ and $\rho_k^{(2)}$, admit the estimates

$$|\rho_k^{(3)}|\leq\frac{2c}{\sqrt{b}\eta^{k+2}},\qquad |\rho_k^{(2)}|\leq\frac{3c^2}{b^{3/2}\eta^{k+2}}. \tag{3.10}$$

We introduce the notation

$$\hat{g}_k(x_1, \ldots, x_k)=E\{\hat{G}(x_1, x_1)\ldots\hat{G}(x_k, x_k)\}. \tag{3.11}$$

Then the obtained result can be written in the form

$$g_k(x_1, \ldots, x_k)=-z^{-1}g_{k-1}(x_2, \ldots, x_k)-\frac{1}{zb}\sum_{|y|\leq m} v^2\left(\frac{x_1-y}{b}\right)\hat{g}_{k+1}(y, x_1, \ldots, x_k)+\rho_k(x_1, \ldots, x_k), \tag{3.12}$$

where

$$\rho_k=\rho_k^{(1)}+\rho_k^{(2)}+\rho_k^{(3)},$$

and by virtue of (3.8) and (3.10)

$$|\rho_k|\leq\frac{c_1k}{\sqrt{b}\eta^{k+2}}, \tag{3.13}$$

where c_1 does not depend on b and is finite for $c < \infty$ (in reality, c_1 depends quadratically on c).

The relation (3.12) differs from (2.10) in that, first, the second term on the right contains $\hat{g}_{k+1}$ and not g_{k+1} and, second, the remainder ρ_k has in accordance with (3.13) the order $O(b^{-1/2})$, and not $O(b^{-1})$ as in (2.11).

With regard to the first difference, the transition from $\hat{g}_{k+1}$ back to g_{k+1} can be made by means of the same identity (3.2), in which G and $\hat{G}$ are interchanged, and by means of the arguments that lead to (3.12) and (3.13). As a result, we have obtained a relation that differs from (3.12) by the replacement of $\hat{g}_{k+1}$ on the right by g_{k+1}. The resulting error has the same estimate as (3.13), i.e., it only changes the constant c_1.

Thus, we have arrived at equations of the form (2.6) and (2.10) for arbitrary bounded $V(x, y)$ that differ from these last obtained for Gaussian $V(x, y)$ only in that the term ρ_k now has the order $O(b^{-1/2})$. However, it is easy to show that the final result expressed by Theorem 2 relies only on the fact that $\|\rho\| \to 0$ for $n, b \to \infty$ and is therefore also valid in the case considered now of $V(x, y)$ satisfying (3.1). Of course, the difference affects the rate of convergence of the corresponding quantities (r_1 to r, $N_n(\lambda)$ to $N(\lambda)$), which we intend to discuss in detail in a following paper.

We now consider arbitrary independent (for $x \leq y$) and symmetric random variables $V(x, y) = V(y, x)$. We introduce the truncated quantities

$$V_c(x, y)=\begin{cases} V(x, y), & |V(x, y)|\leq c,\\ c, & |V(x, y)|>c,\end{cases}$$

and set

$$W_c^{(n)}(x, y)=V_c(x, y)v\left(\frac{x-y}{b}\right),\quad U_c^{(n)}(x, y)=W^{(n)}(x, y)-W_c^{(n)}(x, y),\quad G_c=(W_c^{(n)}-z)^{-1},\quad G=(W^{(n)}-z)^{-1}. \tag{3.14}$$

Then if $r_{n,c} = n^{-1}\,\mathrm{Sp}\, G_c$ is the Stieltjes transform (2.30) of the function (1.3) for $W_c^{(n)}$, then in accordance with the above in this and the previous sections for $\nu = \infty$, fixed c, and $|\,\mathrm{Im}\, z\,| \geq 3V$

$$\lim_{n\to\infty} E\{|r_{n,c}(z)-r_c(z)|^2\}=0, \tag{3.15}$$

where the function $r_c(z)$ is the unique solution of the equation

$$r_c(z)=-(z+V_c^2 r_c(z))^{-1},\qquad V_c^2=\int\limits_{|V|\le c} V^2\,dF(V) \tag{3.16}$$

and $F(V)$ is the distribution function of the random variables $V(x, y)$. It is clear that

$$\lim_{c\to\infty} V_c^2=V^2,$$

where V^2 is defined in (1.2). The "limit", as $c \to \infty$, form of Eq. (3.16) has the form

$$r(z)=-(z+V^2 r(z))^{-1}, \tag{3.17}$$

and it is readily shown that

$$\lim_{c\to\infty}\ \sup_{|\mathrm{Im}\, z|\ge 3V} |r(z)-r_c(z)|=0. \tag{3.18}$$

In addition, on the basis of the resolvent identity (2.3), in which $A + B = W^{(n)}$, $B = U_c^{(n)}$,

$$|n^{-1}\,\mathrm{Sp}\, G-n^{-1}\,\mathrm{Sp}\, G_c|=|n^{-1}\,\mathrm{Sp}\, GU_cG_c|\le\frac{1}{\eta^2}\Big(\frac{1}{n}\sum_{|x|,|y|\le m}|U_c^{(n)}(x,y)|^2\Big)^{1/2},$$

from which we find on the basis of (2.4) and (3.14) that

$$E\{|r_n(z)-r_{n,c}(z)|\}\le\frac{\mathrm{const}}{\eta^2}\Big(\int\limits_{|V|>c} V^2\,dF(V)\Big)^{1/2}, \tag{3.19}$$

where constant is an absolute constant.

Therefore, if $r(z)$ is the solution (1.18) of Eq. (3.17) (or (3.15)), then

$$E\{|r-r_n|\}\le|r-r_c|+E\{|r_c-r_{n,c}|+|r_{n,c}-r_n|\}.$$

Hence, taking into account (3.15), (3.18), (3.19), and (2.4), we obtain the relation (1.14) for arbitrary matrices.

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